Some remarks on Large Deviations

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Conference in memory of
Larry Shepp
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- Need to be set up properly
- Look at three examples

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- Compactification or some control

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- $c(G) = \inf_{f \in G^c} I(f)$

$$I(f) = \sup_{g\psi} \left[\int_0^1 f(t)g(t)dt\psi \right.$$
$$\left. - \frac{1}{2} \int_0^1 \int_0^1 \rho(s,t)g(s)g(t)dsdt\psi \right]$$

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- Do the constants always match?

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- $$P[||X|| \geq C\sqrt{n}] = P[\frac{|X_1 + X_2 + \cdots + X_n|}{\sqrt{n}\psi} \geq C\sqrt{n}]$$
$$\leq P[|X_1| + \cdots |X_n| \geq Cn]$$
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Provided $C > E[|X|]$ and $E[e^{\theta|X|}] < \infty$

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$X\psi= X(1)$. $X(t)$ is a continuos process with independent increments.

- $$\begin{aligned} P[|X(1)| \geq n] &\leq P[\sup_{0\leq t\leq 1} |X(t)| \geq n] \\ &\leq P[\tau_1 + \tau_2 + \ldots + \tau_n \psi\leq 1] \\ &\leq e[E[e\psi^{(\tau_1+\cdots+\tau_n)}]] \\ &= e[E[e\psi^{\tau}]^n] \end{aligned}$$

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- Improves it to get the right constant.

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- $E[e^{\theta|X|}] < \not\infty$ for all $\theta > 0$.
- For a Gaussian this follows from $E[e^{\theta|X|}] < \infty$ for some $\theta > 0$.

Example. Sourav Chatterjee

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$$\binom{n\psi}{r\psi} = \exp[-n[x \log x + (1-x)\log(1-x)] + o(n)]$$

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- For coin tossing with a biased coin

$$I(x) = x \log \frac{x\psi}{p\psi} + (1-x)\log\frac{1-x\psi}{1-p\psi}$$

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- In a given graph $\mathcal{G}$ this may be smaller and the ratio is some fraction

$$r(N,\mathcal{G},\Gamma) \leq 1$$

- Count the number of graphs $\mathcal{G}$ having $N\psi$ vertices with specified values $r(N, \mathcal{G}, \Gamma_i) = r_i\psi$ for a finite number of Γ's.

- Count the number of graphs $\mathcal{G}$ having N vertices with specified values $r(N, \mathcal{G}, \Gamma_i) = r_i$ for a finite number of Γ's.
- Their number is

$$\exp[N^2 J(\Gamma_1, r_1; \ldots; \Gamma_k, r_k) + o(N^2)]$$

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- Out of a total of $2^{\binom{N}{2}}$ possible graphs with $N\psi$ vertices.
- $0 \leq J\psi \leq \frac{1}{2} \log 2$
- expression for $J\psi$

- $0 \leq x, y\psi \leq 1; f(x, y) = f(y, x);\ 0 \leq f\psi \leq 1$

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- $$r(\Gamma, f) = \int_{[0,1]^{\mathcal{V}(\Gamma)}} \prod_{(i,j)\in\mathcal{E}(\Gamma)} f(x_i, x_j) \prod_{i\in\mathcal{V}(\Gamma)} dx_{i\psi}$$

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- $$J = \sup_{\substack{f : r(\Gamma_i, f) \quad r_i \\ 1 \le i \le k}} H(f)$$

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- Consider a "random" graph with $N\psi$ vertices labeled $\{1, 2, \ldots, N\}$. (i, j) is an edge with probability $f(\frac{i}{N}, \psi\frac{j\psi}{N\psi})$.
- The "expected number" of subgraphs Γ can be easily calculated.

- Consider a map ϕ of Γ onto $\{1, 2, \ldots, N\}$.

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- There are $N(N-1)\cdots(N-k+1)$ of them
- The chance that one of them maps edges in Γ to edges in our random graph is

$$\Pi_{(v,v')\in E(\Gamma)} f\left(\frac{\phi(v)}{N}, \frac{\phi(v')}{N}\right)$$

- Ratio of the expected number of subgraphs of type Γ to the number in a complete graph, for large $N \psi$ is

$$r(\Gamma, f) = \int_{[0,1]^{\mathcal{V}(\Gamma)}} \Pi_{(i,j)\in\mathcal{E}(\Gamma)} f(x_i, x_j) \Pi_{i\in\mathcal{V}(\Gamma)} dx_i \psi$$

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Law of large numbers is valid.

- $$w(\mathcal{G}) = \Pi_{(i,j)\in\mathcal{E}(\mathcal{G})} f(\frac{i}{N}, \psi\frac{j\psi}{N\psi}) \Pi_{(i,j)\notin\mathcal{E}(\mathcal{G})} [1 - f(\frac{i}{N}, \psi\frac{j\psi}{N\psi})]$$

$$\sum_{\mathcal{G}\in\mathcal{G}_{N,\epsilon,r_1,r_2,\ldots,r_k}} w(\mathcal{G}) \simeq 1$$

- The typical probability $w(\mathcal{G})$ under the distribution determined by $f\psi$ has the property $\log w(\mathcal{G}) =$

$$\sum_{(i,j)\in\mathcal{E}(\mathcal{G})} \log f(\frac{i}{N},\psi\frac{j\psi}{N\psi}) + \sum_{(i,j)\notin\mathcal{E}(\mathcal{G})} \log[1 - f(\frac{i}{N},\psi\frac{j}{N\psi})]$$

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- The typical probability $w(\mathcal{G})$ under the distribution determined by f has the property $\log w(\mathcal{G}) =$

$$\sum_{(i,j)\in\mathcal{E}(\mathcal{G})} \log f(\frac{i}{N},\frac{j}{N}) + \sum_{(i,j)\notin\mathcal{E}(\mathcal{G})} \log[1 - f(\frac{i}{N},\frac{j}{N})]$$

- $-\frac{N^2}{2}H(f)$
- You must have at least $\exp[\frac{N^2}{2}H(f)]$ graphs.

- $X\psi$

$$\begin{matrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n\psi} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n\psi} \\ \ldots\psi & \cdots & \cdots & \cdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n\psi} \end{matrix}$$

- $k\psi \in \mathcal{K}_{n\psi}$

$x_{1,1}$	$x_{1,2}$	$\cdots$	$x_{1,n\psi}$
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- $\mathcal{K} = \{k : k(x,y), [0,1]^2 \to [0,1]\}$
- $\mathcal{K}_N$: range of $N \times N$ matrices.

- $\mathcal{K} = \{k\not\psi : k(x,y), [0,1]^2 \to [0,1]\}$
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- $\log P[k_{N\psi} \simeq f]$

$$\simeq - I(f)$$

$$= \frac{N^2}{2} \int f \log(2f) + (1-f)\log(2(1-f))dxdy\psi$$

$$= N^2[H(f) - \frac{1}{2}\log 2]$$

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$$J(\Gamma_1, r_1; \ldots; \Gamma_k, r_k)) = \frac{1}{2}\log 2 - \inf_{\substack{k\, : r(\Gamma_i\,, k) \quad r_i \\ 1 \le i \le k}} I(k)$$

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- Strong topology like L_p will be OK.
- No chance. Fluctuations.

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- Enter "cut" topology
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$$\sup_{|\phi|,| \quad | \leq 1} \int [k_1(x,y) - k_2(x,y)]\phi(x) \quad (y) dx dy \psi$$

- $$\sup_{A,B\psi} \int_{A \times B\psi} [k_1(x,y) - k_2(x,y)] dx dy \psi$$

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- Divide by the number in a complete graph.

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- There is s symmetric function $f(x,y)$ on $[0,1]\times[0,1]$ such that
- For any graph Γ with vertices $\mathcal{V}(\Gamma)$ and edges $\mathcal{E}(\Gamma)$

$$r(\Gamma,f)=\int_{[0,1]^{\mathcal{V}(\Gamma)}}\Pi_{(i,j)\in\mathcal{E}(\Gamma)}f(x_i,x_j)\Pi_{i\in\mathcal{V}(\Gamma)}dx_{i\psi}$$

- $r(\Gamma, f) = r(\Gamma, g)$ for all Γ if and only if $f(x, y) = g(\sigma x, \sigma y)$ for some $\sigma\psi \in \mathcal{H}.\psi$

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- Cut topology is the smallest topology on $\mathcal{K}/\mathcal{H}$ that makes $f\psi \mapsto r(\Gamma, f)$ continuous for every Γ.
- This topology works for LLN. $2^{n\psi} \times 2^{n\psi} << \psi 2^{n^2}$

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- It may be possible to prove the large deviation estimate in the topology induced by "cut" topology on $\mathcal{K}/\mathcal{H}$

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- $\log n! = o(n^2)$

Example. Chiranjib Mukherjee

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$$\lambda(V) = \lim_{T\to\infty} \frac{1}{T\psi} \log E[\exp[\int_0^T V(x(s))ds]]$$

$$= \sup_{|f|_2=1} \left[\int V(x)[f(x)]^2 dx - \frac{1}{2}\int |\nabla f|^2 dx\psi \right.$$

$$= \sup_{\substack{f\geq 0 \\ |f|_1 \quad 1}} \left[\int V(x) f(x) dx - \frac{1}{8}\int \frac{|\nabla f|^2}{f} dx\psi \right]$$

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Compactification of $\mathcal{M}(\mathcal{R}^d)/\mathcal{R}^{d\psi}$

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- Completion is compact.

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- a is not determined.

THANK YOU